

# EXAMPLE ON THE APPLICATION OF FORCE AND DISPLACEMENT METHODS ON THE ANALYSIS OF STATICALLY INDETERMINATE FRAMES

## 1.0 Introduction to methods of structural analysis

It is a well known fact that there are two basic methods of structural analysis of indeterminate structures which are;

1. Flexibility methods (also known as force methods, compatibility methods or the methods of consistent deformation) and
2. Displacement methods (also known as stiffness or equilibrium method)

Each method involves the combination of a *particular solution* which is obtained by making the structure statically determinate, and a *complementary solution* in which the effects of each individual modification is assessed. In the force methods, the behaviour of the structure is considered in terms of unknown forces, while in the stiffness method, the behaviour of the structure is considered in terms of unknown displacements. By implication, both analysis methods always involve reducing the structure to a basic system (a determinate system). In the force method, the basic system involves the removal of redundant forces, while the stiffness method involves restraining the joints of the structure against displacement.

Whenever we are using the force method, our basic system is dependent on the degree of static indeterminacy, which a system that must be statically determinate and stable. The choice of redundant constraints to remove is based on experience, geometrical configuration, stability of the structure, and ease of analysis. Degree of static indeterminacy can be computed using;

$$D = (3M + R) - 3N \text{ ----- (1)}$$

Where;

D = Degree of static indeterminacy

M = Number of members

R = Number of constraints (reactions)

N = Number of nodes

Alternatively, the following relation can be used;

$$D = R - 3 - S \text{ ----- (2)}$$

Where;

D = Degree of static indeterminacy

R = Number of constraints (reactions)

S = Number of special conditions (such as an internal hinge)

After selecting a good basic system, we now replace the redundant supports with unit loads and evaluate them one after another. In this step, we calculate the deflection corresponding to each redundant force separately due to applied loading and other redundant forces from force-displacement relations. Deflection due to redundant force cannot be obtained without knowing the magnitude of the

redundant force. Therefore, we apply a unit load in the direction of redundant force and determine the corresponding deflection. Since the principle of superposition is valid in elastic analysis, the deflections due to redundant force can be obtained by multiplying the unknown redundant with the deflection obtained from applying unit value of force. Now, we calculate the total deflection due to the applied loading and the redundant force by applying the principle of superposition which must be compatible with the existing boundary condition.

For more than one set of redundant forces, we construct a set of simultaneous equations with the redundant forces as unknowns and flexibility coefficients as coefficients of the equations. These flexibility coefficients are also called the *influence coefficients*. The total number of equations equals the number of unknown redundant forces. These set of simultaneous equations are called the canonical equations, and they are of the form shown below;

$$\begin{array}{cccccccc} \delta_{11}X_1 + \delta_{12}X_2 + \dots + \delta_{1n}X_n + \Delta_1P & = & 0 \\ \delta_{21}X_1 + \delta_{22}X_2 + \dots + \delta_{2n}X_n + \Delta_2P & = & 0 \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ \delta_{n1}X_1 + \delta_{n2}X_2 + \dots + \delta_{nn}X_n + \Delta_nP & = & 0 \end{array} \quad (3)$$

Where;

$\delta_{ij}$  = deformation at point  $i$  due to a unit load at point  $j$

$X_i$  = Redundant force at point  $i$

$\Delta_iP$  = Deformation at point  $i$  due to externally applied load

$$\text{Mohr's integral } \delta_{ij} = \int_0^l \frac{M\bar{m}}{EI} ds \quad (4)$$

Where;

$\delta_{ij}$  = deformation at point  $i$  due to a unit load at point  $j$

$M$  = Moment due to externally applied load or another redundant load

$\bar{m}$  = Moment due to unit load (virtual load)

## Displacement method

Analysis of statically indeterminate structures by the displacement method in the canonical form begins with determining the degree of kinematical indeterminacy. The fundamental approach in the displacement method is the opposite of force method. First of all, we calculate the *deformations* at the ends of the members and then the internal forces in the members. Thus, the primary unknowns in the displacement method are the *displacements*.

Analysis of any statically indeterminate structure by the displacement method begins with determining the degree of *kinematical* indeterminacy. Generally, the degree of *kinematical* indeterminacy  $n$  of a structure is determined by the formula;

$$n = n_r + n_d \quad (5)$$

Where;

$n_r$  is the number of unknown angles of rotation of the rigid joints of a structure, and

$n_d$  is the number of independent linear displacements of the joints.

In general, the degrees of *kinematical* and static indeterminacy are not equal. The primary system of the displacement method is obtained from the given one by *introducing* additional constraints to prevent rotation of all rigid joints and all independent displacements of various joints. These introduced constraints are shown by the shaded squares and the double lines, respectively.

Primary unknowns  $Z_i$  ( $i = 1, 2, 3, \dots, n$ ) represent *displacements* of introduced constraints (angles of rotations and/or linear displacements of various joints of a frame). The number of primary unknowns,  $n$ , for each structure equals to the degrees of its kinematical indeterminacy.

The canonical equations of the displacement method will be written as follows:

$$\begin{array}{cccccccc} k_{11}Z_1 + k_{12}Z_2 + \dots + k_{1n}Z_n + K_1P & = & 0 \\ k_{21}Z_1 + k_{22}Z_2 + \dots + k_{2n}Z_n + K_2P & = & 0 \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ k_{n1}Z_1 + k_{n2}Z_2 + \dots + k_{nn}Z_n + K_nP & = & 0 \end{array} \quad \text{----- (6)}$$

Where;

$k_{ij}$  = Unit reaction (force or moment) at point  $i$  due to a unit displacement at point  $j$

$X_i$  = Real unknown displacement at point  $i$

$\Delta_i P$  = Reaction at point  $i$  due to externally applied load

### 3.0 Guide to choice of method

It is expected that the reader should by now have known the basic principles underlying both analysis methods. Often times in the classroom, the lecturer often specify the method he wants the students to use, based on the scheme of work. Apart from that, sometimes it is required that manual methods be used to verify some results produced by computer programs, or when you are verifying the results produced using a MATLAB code you have written. The following are simple guides to choice of method.

1. Check the degree of static and kinematic indeterminacy using equations 1 and 5 above. The one that has the less value is usually the most favourable when you are using your classroom non-programmable calculator. However, when you have programs that can help you solve the matrices (e.g. Microsoft excel or MATLAB) such advantage could be written off.
2. Check the loading configuration on the structure. The more complex the loading, the less advantageous force method becomes over the displacement method. This is because in the force method, the graphical method (use of moment diagrams) or the integration method (Mohr's integral) is usually used. The more complex the loading, the more complex the diagrams are expected to be, and of course the more number of sections you are expected to cut in order to generate moment equations for your integration. But if the displacement method is used, all loads are converted to FIXED END MOMENTS and SUPPORT REACTIONS, which is quite handier than the former.
3. When you are seeking to obtain the rotation at the nodes, the displacement method gains more advantage over the force method because the solution of the canonical equation gives the displacement of the structure. However, if force method is used, it is imperative to obtain the final bending moment diagram of the structure, before proceeding. This is also the case when you are looking to obtain the side sway of the frame.
4. Based on the nature of the analysis methods, it is faster to obtain support reactions of frames using force method since the solution of the canonical equations give the solution. However in displacement method, support reactions can be obtained from 1<sup>st</sup> principle after analysis of the bending moments. A variation of this is in the direct stiffness method, which is better handled using a standard computer program. It is not handy for manual calculations at all.
5. The force method handles frames of non-symmetric configuration better than the displacement method when computing manually. For instance, it is easier to handle frames that are made up of slanted members with force method than with displacement method since

nothing changes about the way moment diagrams are computed. When slanted or non-symmetric members are involved in displacement method, the side sway mechanism can get complicated, and errors can creep in.

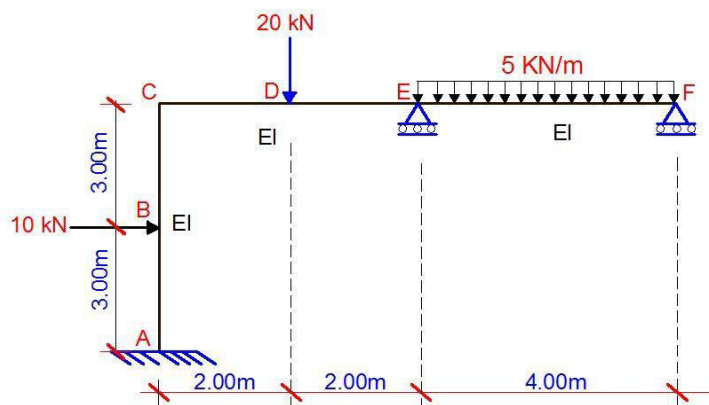
6. When writing structural analysis programs in computer, displacement method (specifically direct stiffness method) is beyond the competition. This simply requires finding a way of obtaining and handling the structure's general stiffness matrix because while if force method is employed, there is a big task of programming how to reduce the structure to basic system (which is a system that is statically determinate and stable). For some structural configurations, it can be a quite daunting task. Even during manual analysis, it is common to see some students getting the basic system wrong. But the basic system in displacement method is easy to conceive.

#### 4.0 Analysis Examples

**Problem 1.0:** A frame is loaded and supported as shown below, obtain and draw the internal stresses diagram (moment, shear and axial) for the externally applied load. Also obtain the side sway of the frame using;

- a) Force method
- b) Displacement method

(EI = Constant)



#### Solution

**NB: DIAGRAMS ARE NOT TO SCALE. READ ASSIGNED VALUES**

(a) By force method;

**Step 1: Determine the degree of static indeterminacy.**

$D = R - E - s$  where;

$D$  = Degree of indeterminacy

$R$  = number of support reactions in the system = 5

$E$  = number of equations of equilibrium = 3

$S$  = any special condition, e.g. internal hinge = 0

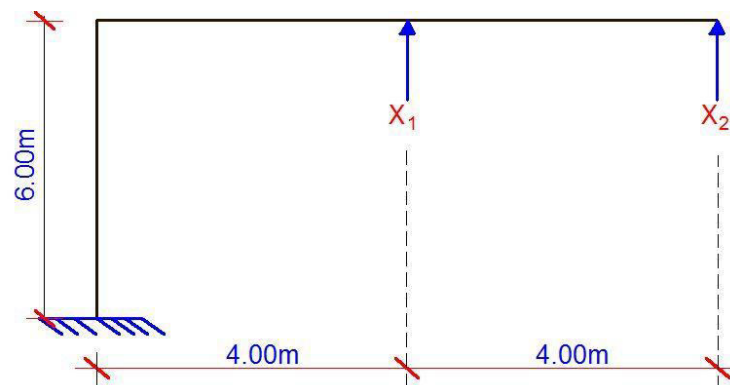
**Therefore,  $N = 5 - 3 - 0 = 2$**

Therefore, the structure is indeterminate to the **2<sup>nd</sup> degree**

And we are expected to remove two redundant supports, and solve a simultaneous equation containing two unknowns in our canonical equation.

### Step 2: Reduce the structure to a basic system

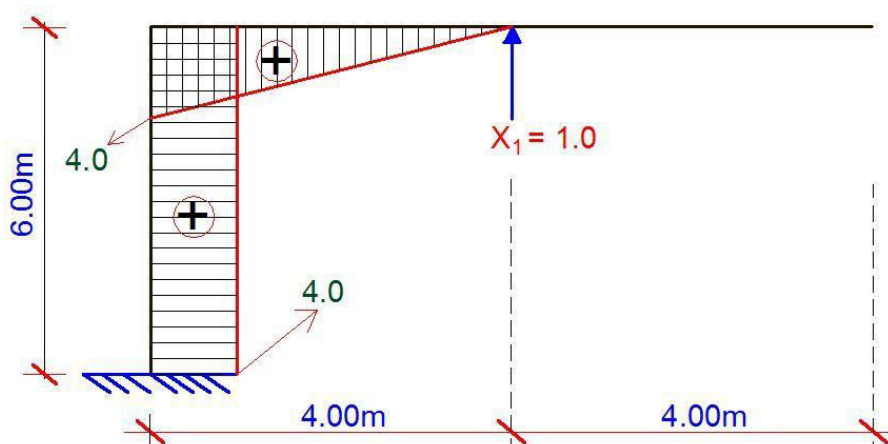
As we realised from the first step, we will need to remove two redundant supports in order to make the structure statically determinate. However, we must ensure that the selected system must be stable. If we remove the roller at supports E and F, we will obtain a good basic system. So in this example, we are removing the reactive forces at supports E and F and replacing them  $X_1$  and  $X_2$  which will be assigned unit values. The basic system is shown below.



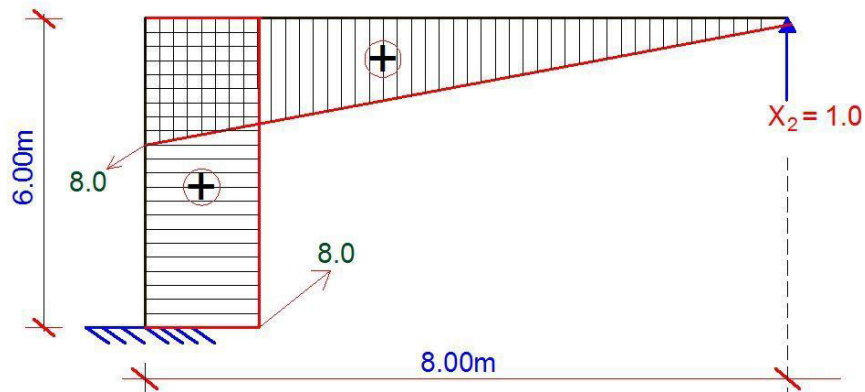
### Step 3: Analysis of the various load cases ( $X_1$ , $X_2$ and externally applied load)

The values  $X_1$  and  $X_2$  represent the redundant forces that have been removed from the system, and they will be assigned unit values in order for us to progress in our analysis. We are going to treat each of them as independent load cases on the structure. Our main interest in this discuss will be on the moment diagrams that they produce. You are expected to be familiar with the analysis of statically determinate frames by now.

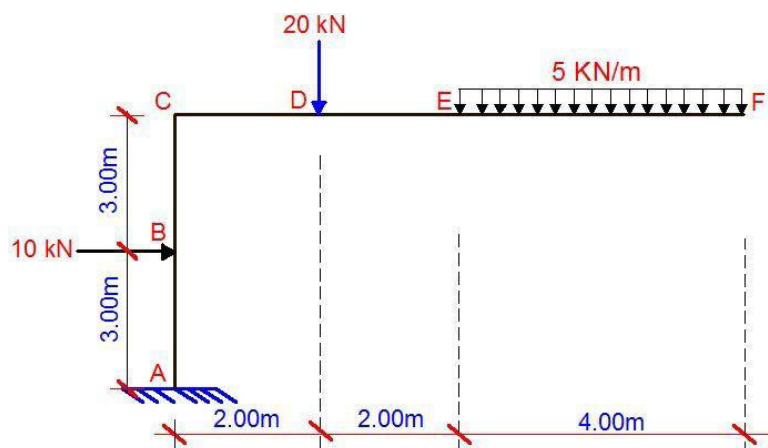
#### Case 1 ( $X_1 = 1.0$ , $X_2 = 0$ )



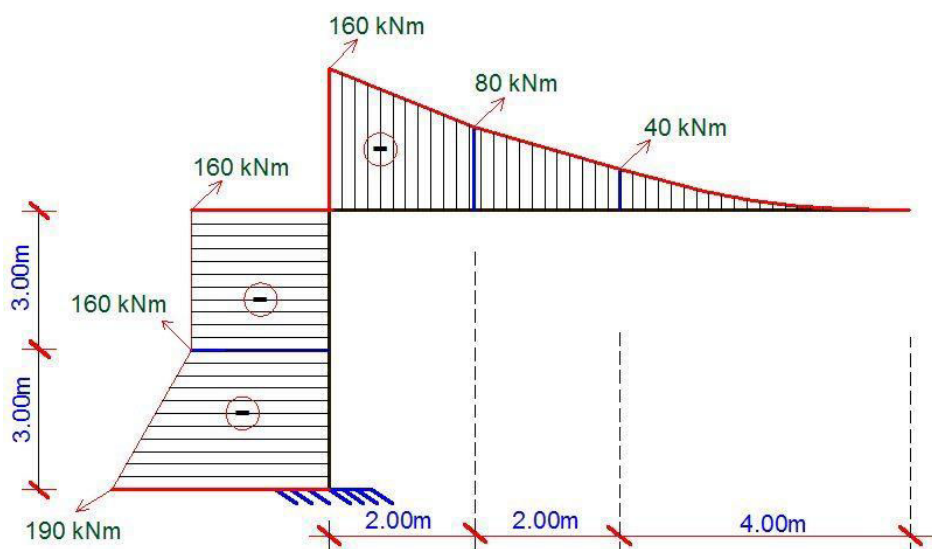
Case 2 ( $X_1 = 0$ ,  $X_2 = 1.0$ )



Loading the basic system with externally applied load

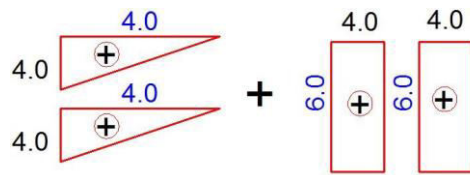


The resulting bending moment diagram due to external load is given below;



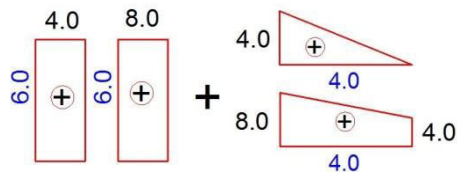
#### Step 4: Computation of influence coefficients

$\delta_{11}$  = Deflection at point 1 due to unit load at point 1



$$\delta_{11} = \left(\frac{1}{3} \times 4 \times 4 \times 4\right) + (4 \times 4 \times 6) = 117.333$$

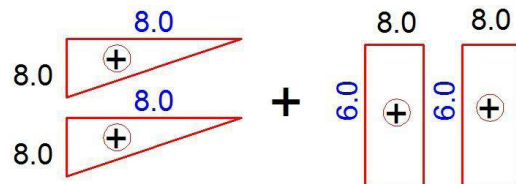
$\delta_{12} = \delta_{21}$  Deflection at point 1 due to unit load at point 2



$$\delta_{12} = (4 \times 8 \times 6) + \frac{1}{6} \times [4 \times ((2 \times 8) + 4) \times 4]$$

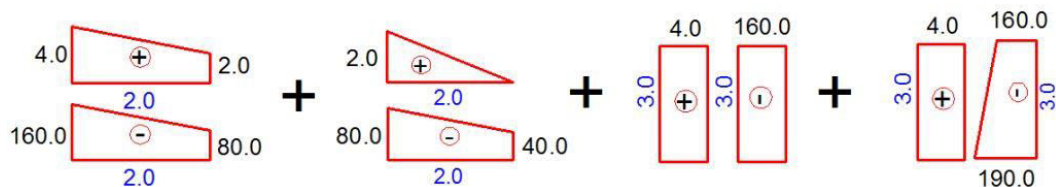
$$\delta_{12} = \delta_{21} = 192 + 53.333 = 245.333$$

$\delta_{22}$  = Deflection at point 2 due to unit load at point 2



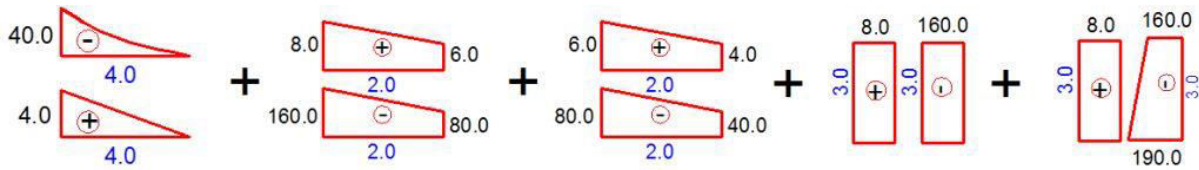
$$\delta_{22} = \left(\frac{1}{3} \times 8 \times 8 \times 8\right) + (8 \times 8 \times 6) = 554.667$$

$\Delta_1 P$ : Deformation at point 1 due to externally applied load



$$\Delta_1 P = -\frac{1}{6} \times 2 [160((2 \times 4) + 2) + 80((2 \times 2) + 4)] - \frac{1}{6} \times [2 \times ((2 \times 80) + 40) \times 2] - (40 \times 160 \times 3) - \frac{1}{2} \times [4 \times (160 + 190) \times 3] = -4900$$

### $\Delta_2 P$ : Deformation at point 2 due to externally applied load



$$\Delta_2 P = -\left(\frac{1}{4} \times 40 \times 4 \times 4\right) - \frac{1}{6} \times 2 \left[160((2 \times 8) + 6) + 80((2 \times 6) + 8)\right] - \frac{1}{6} \times 2 \left[80((2 \times 6) + 4) + 40((2 \times 4) + 6)\right] - (8 \times 160 \times 3) - \frac{1}{2} \times [8 \times (160 + 190) \times 3] = -10520$$

### Step 5: Canonical equations

The appropriate canonical equation is

$$\delta_{11}X_1 + \delta_{12}X_2 + \Delta_1 P = 0$$

$$\delta_{21}X_1 + \delta_{22}X_2 + \Delta_2 P = 0$$

$$117.333X_1 + 245.333X_2 = 4900$$

$$245.333X_1 + 554.667X_2 = 10520$$

Solving simultaneously,

$$X_1 = 27.995 \text{ KN}$$

$$X_2 = 6.584 \text{ KN}$$

Note now that the values  $X_1$  and  $X_2$  represent the actual vertical support reactions occurring at roller supports E and F. Since we have known these values, the structure has become determinate, and we can analyse it by the law of statics, or progress using the force method. In this example, we are using the force method to obtain the final moment diagram, and not the shear and axial forces (see further examples for shear and axial).

### Step 6: Final internal stresses (moment only in this example)

$$M_F = \bar{M}_1 X_1 + \bar{M}_2 X_2 + M_0$$

$$M_A = (27.995 \times 4) + (6.584 \times 8) - 190 = -25.348 \text{ KN.m}$$

$$M_B - M_C^B = (27.995 \times 4) + (6.584 \times 8) - 160 = 4.652 \text{ KN.m}$$

$$M_C^R = (27.995 \times 4) + (6.584 \times 8) - 160 = 4.652 \text{ KN.m}$$

$$M_D = (27.995 \times 2) + (6.584 \times 6) - 80 = 15.494 \text{ KN.m}$$

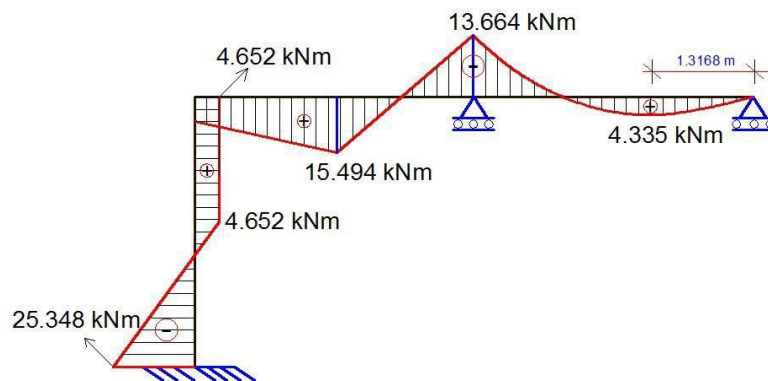
$$M_E = (27.995 \times 0) + (6.584 \times 4) - 40 = -13.664 \text{ KN.m}$$

$$M_F = 0$$

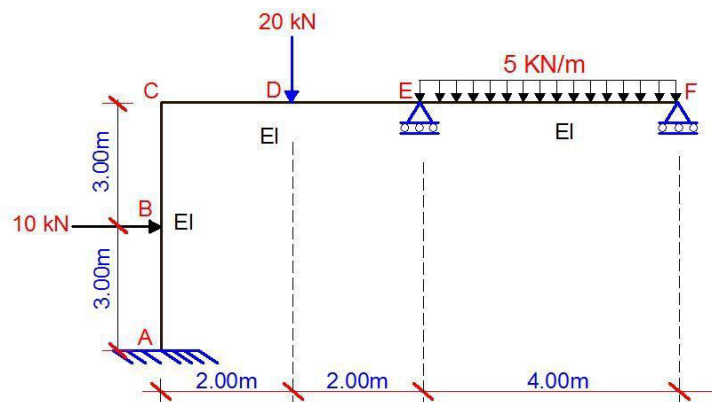
Determine the maximum moment on span E-F using any method of your choice



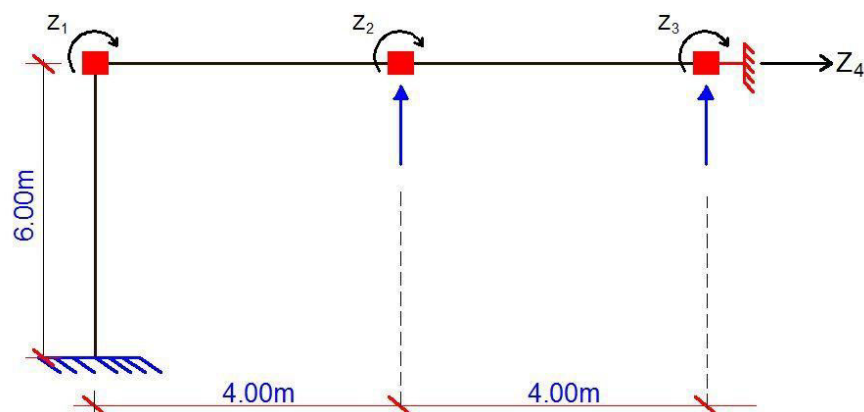
**Step 7: Plot the final internal stresses diagram (moment only in this example)**



**(a) By displacement method;**

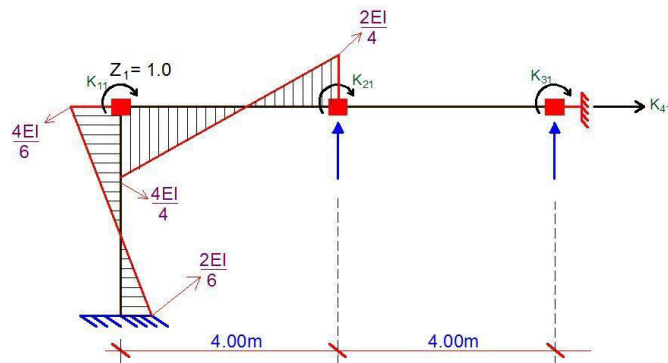


To analyse the same frame by displacement method, we make the structure kinematically determinate. To start with, we consider the number of unknown nodal displacements that the structure can undergo. A little consideration will show that the structure is kinematically indeterminate to the 4<sup>th</sup> degree. These unknown displacements are the rotations at nodes C, E, G, and the lateral translations at F (side sway). To make the structure kinematically determinate, we rigidly fix up nodes C (labelled 1), E (labelled 2), and F (labelled 3), and also put a constraint at nodes F (labelled 4) to prevent lateral displacement. This is shown in the figure below;



### Analysis of case 1; $Z_1 = 1.0$

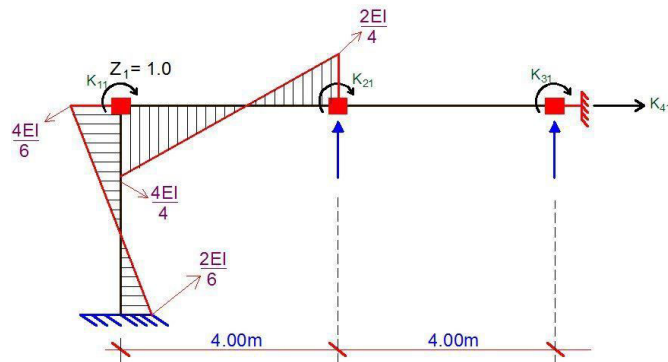
Applying a unit rotation at point 1



$$K_{11} = \frac{4EI}{6} + \frac{2EI}{8} = 1.667EI; K_{21} = \frac{2EI}{4} = 0.5EI; K_{31} = 0; K_{41} = \frac{-6EI}{6^2} = -0.167EI;$$

### Analysis of case 2; $Z_2 = 1.0$

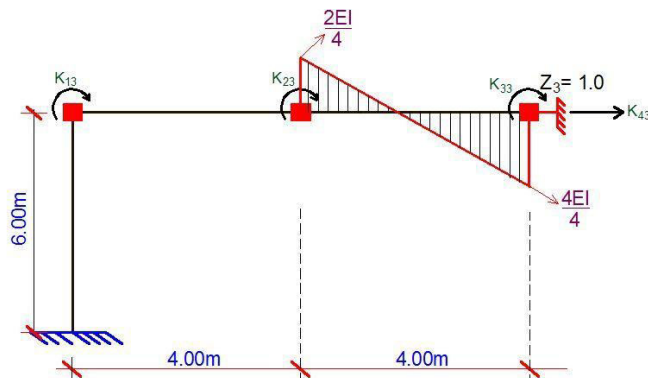
Applying a unit rotation at point 2



$$K_{12} = \frac{2EI}{4} = 0.5EI; K_{22} = \frac{4EI}{4} + \frac{3EI}{4} = 1.75EI; K_{32} = 0; K_{42} = 0$$

### Analysis of case 3; $Z_3 = 1.0$

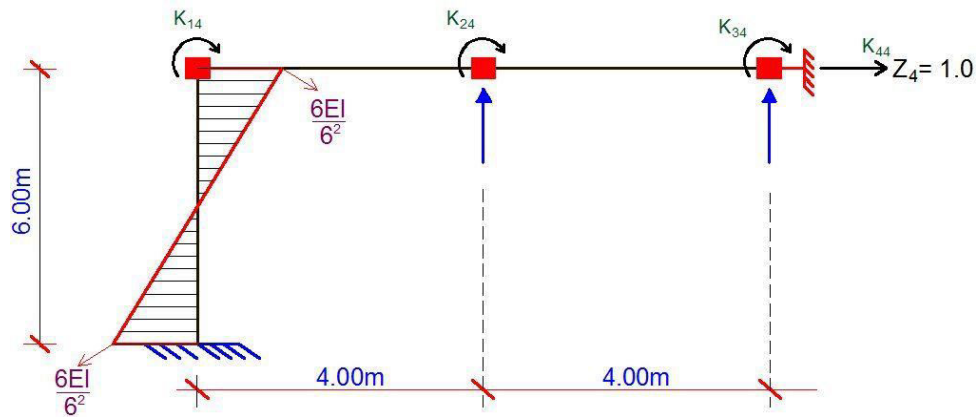
Applying a unit rotation at point 3



$$K_{13} = 0; K_{23} = \frac{2EI}{4} = 0.5EI; K_{33} = \frac{4EI}{4} = EI; K_{43} = 0$$

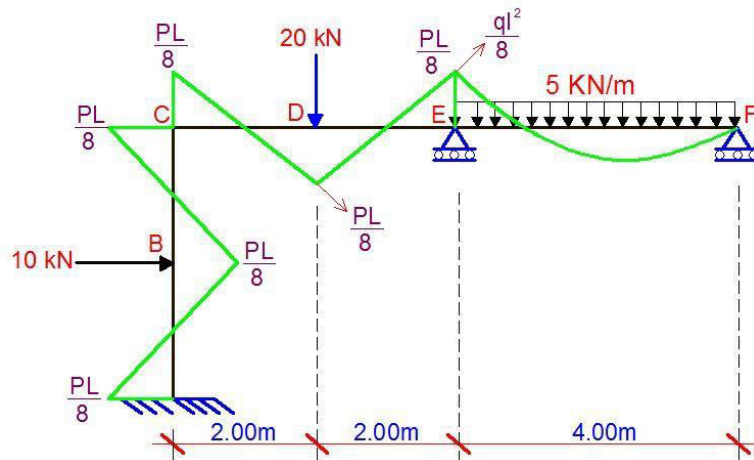
#### Analysis of case 4; $Z_4 = 1.0$

Applying a unit lateral displacement at point 4



$$K_{14} = \frac{-6EI}{6^2} = -0.167EI; K_{24} = 0; K_{34} = 0; K_{44} = \frac{12EI}{6^3} = 0.0555EI$$

#### External load



$$K_{1P} = \frac{PL}{8} - \frac{PL}{8} = \frac{10 \times 6}{8} - \frac{20 \times 4}{8} = -2.5 \text{ KNm}$$

$$K_{2P} = \frac{PL}{8} - \frac{ql^2}{8} = \frac{20 \times 4}{8} - \frac{5 \times 4^2}{8} = 0$$

$$K_{30} = 0$$

$$K_{40} \text{ (reaction from horizontal load)} = -\frac{P}{2} = -\frac{10}{2} = -5 \text{ KN}$$

The general canonical equation is given by;

$$\begin{aligned} k_{11}Z_1 + k_{12}Z_2 + \dots + k_{1n}Z_n + K_{1P} &= 0 \\ k_{21}Z_1 + k_{22}Z_2 + \dots + k_{2n}Z_n + K_{2P} &= 0 \\ \vdots & \\ k_{n1}Z_1 + k_{n2}Z_2 + \dots + k_{nn}Z_n + K_{nP} &= 0 \end{aligned} \quad \text{----- (10)}$$

Hence,

$$\begin{bmatrix} 1.667 & 0.5 & 0 & -0.167 \\ 0.5 & 1.75 & 0 & 0 \\ 0 & 0.5 & 1 & 0 \\ -0.167 & 0 & 0 & 0.0555 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 0 \\ 0 \\ 5 \end{bmatrix}$$

On solving;

$$Z_1 = \frac{17.0930}{EI} \text{ radians; } Z_2 = \frac{-4.8837}{EI} \text{ radians; } Z_3 = \frac{2.4419}{EI} \text{ radians; } Z_4 = \frac{141.2791}{EI} \text{ metres;}$$

### 5.7 Final Moment Values:

$$M_{\text{final}} = M_0 + M_1Z_1 + M_2Z_2 + M_3Z_3 + M_4Z_4 \text{----- (10)}$$

$$M_A = -7.5 + \frac{17.0930}{EI} \left( \frac{2EI}{6} \right) + 0 + 0 - \frac{141.2791}{EI} \left( \frac{6EI}{6^2} \right) = -25.348 \text{ KNm}$$

$$M_C^B = +7.5 + \frac{17.0930}{EI} \left( \frac{4EI}{6} \right) + 0 + 0 - \frac{141.2791}{EI} \left( \frac{6EI}{6^2} \right) = -4.651 \text{ KNm}$$

$$M_C^R = -10 + \frac{17.0930}{EI} \left( \frac{4EI}{4} \right) - \frac{4.8837}{EI} \left( \frac{2EI}{4} \right) + 0 + 0 = 4.651 \text{ KNm}$$

$$M_E^L = +10 + \frac{17.0930}{EI} \left( \frac{2EI}{4} \right) - \frac{4.8837}{EI} \left( \frac{4EI}{4} \right) + 0 + 0 = 13.663 \text{ KNm}$$

$$M_E^R = -10 + 0 - \frac{4.8837}{EI} \left( \frac{3EI}{4} \right) + \frac{2.4419}{EI} \left( \frac{2EI}{4} \right) + 0 = -13.663 \text{ KNm}$$

You can compare these answers with answers from force method.